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Space curves not contained in low degree surfaces in positive characteristic

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Abstract. Let $C \subset \mathbf{P}^3$ be an integral projective curve not contained in a quadric surface. Set $d := \deg(C), g := p_a(C)$,

$$\pi_1(d,3) := \begin{cases} d^2/6 - d/2 + 1 & \text{if } d/3 \in \mathbf{Z} \\ d^2/6 - d/2 + 1/3 & \text{if } d/3 \notin \mathbf{Z} \end{cases}$$

Here we prove in arbitrary characteristic that $g \le \pi_1(d,3)$ if $d \ge 25$.

Keywords: integral projective curve; singular space curve; arithmetic genus; quadric surface; plane section: hyperplane section: Hilbert function.

MSC 2000 classification: 14H50

Introduction

In characteristic zero using heavily the Uniform Position Principle D. Eisenbud and J. Harris proved the following result ([4, Th. 3.13]) (classically due to Halphen).

Theorem 1. *Let $C \subset \mathbf{P}^3$ be an integral projective curve not contained in a quadric surface. Set $d := \deg(C)$, $g := p_a(C)$,*

$$\pi_1(d,3) := \begin{cases} d^2/6 - d/2 + 1 & \text{if } d/3 \in \mathbf{Z} \\ d^2/6 - d/2 + 1/3 & \text{if } d/3 \notin \mathbf{Z} \end{cases}$$

Assume $d \ge 25$. Then $g \le \pi_1(d,3)$.

The main aim of this paper is to prove Theorem 1 in arbitrary characteristic. If we may apply the Uniform Position Principle to the generic hyperplane section of C, then the proof of [4, Th. 3.13], works verbatim. By [5, Cor. 1.8], if the monodromy group of the generic hyperplane section, G, of C contains the alternating group A_d, then the Uniform Position Principle holds for the general plane section of C.

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Unfortunately, we are interested in a case which (if it exists) has as Galois group of the generic hyperplane section the third class of groups in the statement of [5, Th. 3.4].

In the last part of section two we will prove (in arbitrary characteristic) the following result proved by J.Harris in characteristic zero ([3]).

Proposition 1. *Fix integers d, k with $k \geq 2$ and $d > k(k-1)$. Set $n := [(d-1)/k]+1$, $\varepsilon := nk-d$ and $\pi(d,k) := d^2/2k+(k-4)d/2+1-\varepsilon(k-\varepsilon-1+(\varepsilon/k))$. Let $C \subset \mathbf{P}^3$ an integral curve with $\deg(C) = d$ and C contained in an integral surface of degree k. Then $p_a(C) \leq \pi(d,k)$. Notice that the integer ε in the statement of Prop. 1 is the unique integer with $0 \leq \varepsilon \leq k-1$ and $\varepsilon \equiv -d \bmod (k)$.*

1 The proofs

Now we will prove Theorem 1 and Proposition 1.

Proof of Theorem 1. Let $H \subset \mathbf{P}^3$ be a general plane. Set $Z := C \cap H$. Let G be the monodromy group of the generic plane section of C. By [5, Cor. 1.6], and its proof, G is 2-transitive and it is 3-transitive if and only if the general secant line to C intersects C only at two points. By Castelnuovo method ([4, Ch. III]) to find a "good" upper bound for g it is sufficient to find a "good" upper bound for all integers $h(Z,t) := h^0(H, \mathbf{O}_Z(t)) - h^0(H, \mathbf{I}_Z(t))$. We distinguish 3 cases and several subcases.

Case (a) . Here we assume that for a general secant line D of C we have $z := card(D \cap C) \geq 3$. By [1] Z is in linear semi-general position, *i. e.* for any two points P, Q of Z the line $\langle P, Q\rangle$ contains exactly z points of Z. Fix any $P \in Z$ and any line $D \subset H$ with D spanned by $Z \cap D$ and $P \notin D$. Thus $card(Z \cap D) = z$. Let $\pi : \mathbf{P}^2 \setminus \{P\} \to D$ be the linear projection. Set $y := card(\pi(Z \setminus \{P\}))$. For every line D' with $P \notin D'$ and D' spanned by $D' \cap Z$, $\pi \mid D'$ induces a bijection of $D' \cap Z$ with $\pi(D' \cap \mathrm{Z})$. Thus $y \geq z$. For every $Q \in \pi(Z \setminus \{P\})$ the line $\langle P, Q\rangle$ contains exactly z points of Z and any two such lines intersect only at $\{P\}$. Thus $d-1 = (z-1)y$.

We claim that for every integer $t \geq z+y-1$ we have $h^1(H, \mathbf{I}_Z(t)) = 0$. Set $\{Q_1, \dots, Q_y\} := \pi(Z \setminus \{P\})$, $D_j := < \{P, Q_j\} >$ and $S_j := Z \cap D_j$. Since $D_j \cong \mathbf{P}^1$ and $card(S_j) = z$, for every integer $u \geq z-1$ we have $h^1(D_j, \mathbf{O}_{D_j}(u)(-S_j)) = 0$. Thus if $t \geq z+y-1$ we obtain $h^1(H, \mathbf{I}_Z(t)) \leq h^1(H, \mathbf{I}_{Z \setminus S_y}(t-1)) \leq h^1(H, \mathbf{I}_{Z \setminus (S_y \cup S_{y-1})}(t-2)) \leq \cdots \leq h^1(H, \mathbf{I}_{D_1 \setminus \{P\}}(t-y+1)) = 0$, proving the claim.

By the definition of z every homogeneous form on H with degree $z-1$ vanishing at Z vanishes on each D_j. There are y distinct lines D_j, $1 \leq j \leq y$, and $y \geq z$.

Thus we have $h^0(H, \mathbf{I}_Z(z-1)) = 0$. Now we will check that if $z \le t \le z+y-2$, then $h(Z,t) \ge z + (t-z)(z-1)$. To check this inequality we may use induction on t because the case $t = z$ follows from $h^1(D_1, \mathbf{O}_{D_1}(z)(-S_1)) = 0$. Set $\Sigma := S_1 \cup \ldots \cup S_{t-z}$ and $\Delta := \Sigma \cup S_{t-z+1}$. Since $h^1(D_{t-z+1}, \mathbf{O}_{D_{t-z+1}}(t)(-S_{t-z+1})) = 0$, we obtain $h(Z,t) \ge h(\Delta, t) \ge h(\Sigma \setminus \{P\}, t-1) + z \ge h(\Sigma, t-1) + z - 1$ and hence we conclude by induction on t. By using all the inequalities for the integers $h(Z,t)$ obtained up to now we obtain $g \le \pi_1(d,3)$ at least if $z \ge 4$.

Now assume $z = 3$. To win it would be sufficient to obtain $h(Z,t) \ge \min\{d, 3t\}$. We have $h(Z,1) = 3$. Fix an integer $t \ge 2$ and assume $h(Z,t-1) \ge \min\{d, 3t-3\}$. Take $S \subset Z$ with $card(S) = h(Z,t-1)$ and set $W := Z \setminus S$. Assume the existence of a line D with $card(D \cap W) \ge 3$. Since $z = 3$ this implies $card(D \cap W) = 3$ and $D \cap S = \emptyset$. Since $h^1(D, \mathbf{O}_D(t)(-(D \cap W))) = 0$, we obtain $h(Z,t) \ge h(S \cup (D \cap W), t) \ge h(S, t-1) + 3$, as wanted.

Now assume that there is no such line. First assume the existence of a conic A with $w := card(A \cap W) \ge 6$. Since W has no trisecant line, A is smooth. We would like to imitate the proofs of $Case(b)$ below, but there are the following differences. First of all, G does not act as the permutation group on W. Furthermore, if $w = card(W)$ and C is linearly normal we cannot use the exact sequence (1) below to conclude. We have $h^1(A, \mathbf{O}_A(t)(-(A \cap W))) = \max\{0, 2t+1-w\}$. If $w \ge 7$ we just take $W' \subseteq W$ with $card(W') = 7$ and any two points of it to obtain $h(S \cup W', t) \ge h(S, t-1) + 2$ and $h(S \cup W', t+1) \ge h(S, t-1) + 7$ and then continue with $S \cup W'$ instead of S; these inequalities are strong enough to obtain that the contribution of $h(Z,t)$ and $h(Z,t+1)$ to the upper bound for g obtained using Castelnuovo method because $2 + 7 = 3 + 6$.

Now assume $w = 6$. If there is an another conic, A', with card$(A' \cap W) \ge 7$,we use A'. Hence from now on we may assume that for every conic A we have card$(A \cap W) \le 6$. We apply the proof of $Case(c)$ without making any mention of the Galois group G; in subcases $(c2)$ and $(c3)$ we will never mention G; in subcase $(c1)$ just note that if we have the union,S,of different points on an irreducible plane cubic $A(S)$ with Sing$(A(S)) \neq \emptyset$, either $S \subset A(S)_{reg}$ or S is not the complete intersection of $A(S)$ with another cubic, because any such complete intersection either does not contain the singular point of $A(S)$ or it has at least multiplicity two at the singular point of $A(S)$.

Case (b) . Here we assume that for a general secant line D of C we have card$(D \cap C) = 2$ (*i. e.* that no 3 points of Z are collinear) but that there is a conic $A \subset H$ with card$(A \cap Z) \ge 6$.

(b1) . First assume $Z \subset A$. If C is linearly normal, then the exact sequence

$$0 \to \mathbf{I}_C(1) \to \mathbf{I}_C(2) \to \mathbf{I}_{Z,H}(2) \to 0 \tag{1}$$

gives $h^0(\mathbf{P}^3, \mathbf{I}_C(2)) \neq 0$, contradicting our assumptions. If C is not linearly normal, then C is an isomorphic linear projection of an irreducible non-degenerate curve $Y \subset \mathbf{P}^4$ with $\deg(Y) = d$. Write $d = 3m + \alpha$ with $0 \leq \alpha \leq 2$. We claim that $p_a(Y) \leq 3m(m-1)/2 + m\alpha$.

To check the claim we distinguish two cases: the generic hyperplane section of Y is in linearly general position or not. In the first case the claim follows from the classical Castelnuovo method. In the second case we are in the case studied in [2] and in particular if $d \geq 25$ we have $d = 2^k$ for some integer k and Y is strange; hence C is strange; but we will not use these informations on d and C.

To check the claim in this case we use the method of part (a). Take a general hyperplane M of $\mathbf{P}^4$ and set $W := Y \cap M$. We need to study the function $h(W,t)$. As in part (a) to check the claim it is sufficient to check that $h(W,t) \geq h(W,t-1) + 3$ if $h^0(M, \mathbf{I}_W(t-1)) \geq 3$ and that $h(W,t) = d$ if $h^0(M, \mathbf{I}_W(t-1)) \leq 2$. Let m be the number of points of W contained in a plane of M spanned by points of W; this number does not depend on the choice of the plane (linear semi-uniform position introduced in [1]). By assumption we have $m \geq 4$.

First assume $m \geq 5$. Take a general $P \in Y_{reg}$. By the generality of P we may take M with $P \in M$. Let $C' \subset \mathbf{P}^3$ be the image of Y under the projection of Y from P. Since a general secant line to Y is not a trisecant line and P is general, C' is birational to Y, $\deg(C') = d-1$ and there is a birational morphism $Y \to C'$. Thus $p_a(Y) \leq p_a(C')$ and it is sufficient to check that $p_a(C') \leq \pi_1(d,3)$. Notice that C' fits in the case considered in part (a) of the proof with $m-1$ as integer z: the image of a plane through P and 2 other general points of Y is a general secant line to C'.

Now assume $m = 4$. Any three points of W span a plane of M and any two planes A, A' of M containing at least 3 points of W have $\mathrm{card}(A \cap A') \leq 2$. Notice that any 4 non-collinear points of a plane impose independent conditions to curves of degree at least 2. Thus we obtain $h(W,t) \geq \min\{d, h(W,t-1)+t\} \geq \min\{d, 3t\}$ (induction on t for the last inequality) with strict inequality for $t = 2$ and 3. Thus in this subcase we obtain $p_a(Y) < 3m(m-1)/2 + m\alpha$. Since $Y \cong C$, the claim concludes the case $Z \subset A$.

(b2) . Now assume $w := \mathrm{card}(A \cap Z) < d$. Since we are not in $Case(a)$, any 3 points of Z span $H([1])$. Thus A is irreducible.

To obtain $g \leq \pi_1(d,3)$ it would be sufficient to show that $h(Z,t) \geq \min\{d, 3t\}$ for every integer t. We have $h(Z,1) = 3$ and $h(Z,2) = 6$. We fix an integer $t \geq 3$ and $S \subseteq A \cap Z$ with $\mathrm{card}(S) = 6$. Since $h^1(A, \mathbf{O}_A(t)(-S)) = 0$, we have $h(Z,t) \geq h(Z \setminus S, t-2) + 6$. If there is a conic B with $\mathrm{card}(B \cap (Z \setminus S)) \geq 6$, we use B in a similar way to show that $h(Z,t) \geq h(Z \setminus (S \cup S'), t-4) + 12$ with $S' \subseteq B \cap (Z \setminus S)$ and $\mathrm{card}(S') = 6$. And so on until we obtain a subset S'' of Z with $e := \mathrm{card}(S'')/6 \in \mathbf{N}$, S'' contained in e conics, $h(Z,t) \geq h(Z \setminus S'', t-2e) + 6e$, but every conic contains at most 5 points of $Z \setminus S''$. If $\mathrm{card}(Z \setminus S'') \geq 9$ we use the same construction using irreducible cubics instead of irreducible conics (see subcase $(c3)$ below).

Now we assume $\mathrm{card}(Z \setminus S'') \leq 8$. First assume $w \geq 7$. Taking instead of S a subset S' with $\mathrm{card}(S') = 7$ we obtain $h(Z,t) \geq h(Z \setminus S', t-2) + 7$. Then using $e-1$ conics if $e \geq 2$ we obtain $h(Z, 2e) \geq 6e+2$. This is sufficient to conclude because in this case it is sufficient to get $h(Z, 2e+1) \geq \min\{d, 6e+4\}$ and $h(Z, 2e+2) \geq \min\{d, 6e+6\}$; the first inequality is obtained using a line and the second one using two conics.

Now assume $w = 6$. Here we pass directly to $Case(c)$. It is just to handle this subcase with $w = 6$ that we allow in $Case(c)$ the existence of a conic containing 6 points of Z.

Case (c) . Here we assume that no 3 points of Z are collinear and that there is no conic containing at least 7 points of Z.

(c1) . Here we assume that no plane cubic contains at least 10 points of Z. We have $h(Z,t) = 0$ for $t \leq \min\{3, (d-1)/3\}$.

Fix an integer $t \geq 3$ and any subset of Z with $\mathrm{card}(S) = 9$. There is a unique cubic curve, $A(S)$, containing S. By assumption $(c1)$ we have $A(S) \cap Z = S$. Since the monodromy group G of the generic hyperplane section is transitive and S is unique, we have $\mathrm{Sing}(A(S)) \cap S = \emptyset$. Hence $h^0(A(S), \mathbf{O}_{A(S)}(t)(-S)) = \deg(\mathbf{O}_{A(S)}(t)(-S)) = 3t-9$ if either $\deg(\mathbf{O}_{A(S)}(t)(-S)) > 0$ (i.e. $t \geq 4$) or $\deg(\mathbf{O}_{A(S)}(t)(-S)) = 0$ (i.e. $t = 3$) and $\mathbf{O}_{A(S)}(t)(-S))$ not trivial and $h^0(A(S), \mathbf{O}_{A(S)}(t)(-S)) = 1$ if $\mathbf{O}_{A(S)}(t)(-S))$ is trivial. We obtain

$$h^0(H, \mathbf{I}_Z(t)) \leq h^0(H, \mathbf{I}_{Z \setminus S}(t-3)) + h^0(A(S),$$
$$\mathbf{O}_{A(S)}(t)(-S)) = h^0(H, \mathbf{I}_{Z \setminus S}(t-3)) + 3t - 9 + \psi$$

with $\psi = 0$ if $t \geq 4$ and $0 \leq \psi \leq 1$ if $t = 3$.

Then if $d \geq 18$ and $t \geq 6$ we continue taking any subset S' of $Z \setminus S$ with $\mathrm{card}(S') = 9$ and the unique plane cubic $A(S')$ with $S' \subset A(S')$. By the assumption $(c1)$ we have $A(S') \cap Z = S'$. Fix a general $\Sigma \subset A(S')$ with $\mathrm{card}(\Sigma) = h^0(A(S'), \mathbf{O}_{A(S')}(t-3)(-S'))$. Every curve $F \subset H$ with $\deg(F) = t-3$ and $S' \cup \Sigma \subset F$ contains $A(S')$. Hence we obtain $h^0(H, \mathbf{I}_{Z \setminus S}(t-3)) \leq h^0(H, \mathbf{I}_{Z \setminus (S \cup S')}(t-6)) + h^0(A(S'), \mathbf{O}_{A(S')}(t-3)(-S'))$. We have $h^0(A(S'), \mathbf{O}_{A(S')}(t-3)(-S')) = 3(t-3) - 9$ unless $t = 6$ and S' is the complete intersection of S' with a cubic curve.

In the latter case we have $h^0(S', \mathbf{O}_{S'}(t-3)(-S')) = 3(t-3) - 0$.
And so on: we continue as in the classical Castelnuovo method using plane cubics through 9 points of Z instead of lines through 2 points of Z. If $t \equiv 1 (resp. 2) \bmod(3)$, then in the last step instead of a plane cubic we use a line (resp. a smooth conic). In this way we obtain $g \leq \pi_1(d, 3)$.

(c2) . Here we assume that Z is contained in a plane cubic, T. Since Z does not contains 7 points on a conic or 3 collinear points, T is irreducible. Hence we may apply verbatim the proof of [4, p. 96], and obtain $g \le \pi_1(d,3)$.

(c3) . Here we assume that Z is not contained in a plane cubic.

Since $h^0(H, \mathbf{O}_H(3)) = 10$ this implies $d \ge 10$. By our assumption there are 10 points of Z not contained in any cubic. Thus there is $S \subset Z$ with $\mathrm{card}(S) = 9$ and a unique cubic $A(S)$ containing S. Again, we obtain $h^0(H, \mathbf{I}_Z(t)) \ge h^0(H, \mathbf{I}_{Z\setminus S}(t-3)) + h^0(A(S), \mathbf{O}_{A(S)}(t)(-S)) = h^0(H, \mathbf{I}_{Z\setminus S}(t-3)) + 3t-9$. $\boxed{QED}$

Proof of Proposition 1. The proof of this result given in [3, §1], in the case of characteristic zero, works verbatim except the proof of a lemma of Gieseker (see [3, p.194]). To extend to positive characteristic the proof given there just use that $\mathbf{P}^1$ is irreducible and hence that by [5, Cor. 1.6], for any base point free linear system on $\mathbf{P}^1$ the monodromy of a generic hyperplane section is transitive ([5, Cor. 1.6]). Alternatively, a far stronger form of this lemma is proved in arbitrary characteristic in [6], Th. 1 and Th. 2. $\boxed{QED}$

We do not know if Theorem 1 holds for low d. We did not checked it case by case because the original question posed to us by G. Korchamaros was for $d = 2^f + 1$ with $f \ge 3$ in characteristic two and the missing case $d = 17$ fits in $case(a)$ (subcase $z = 3$) and $case(c)$ which are the worst cases for our approach. The bound on g given by Theorem 1 is sharp in arbitrary characteristic for curves contained in cubic surfaces. We do not know how much it may be improved assuming C not contained in a cubic surface but without assuming (as in Proposition 1) that C is contained in a low degree surface.
In several subcases the proof of Theorem 1 gives far better bounds for g. The difficult cases (for our method) are $case(a)$ (subcase $z = 3$) and $case(c)$ (or $case(b)$ with $w = 6$) and in these cases we do not know how to improve our bound.

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